9: Relaxation of nuclear magnetization

1. How is the MR signal detected?
2. What is the quantum-mechanical equivalent of the rotating frame?
3. What is the rotating frame description good for?
4. How can the return of the magnetization to thermodynamic equilibrium described?
5. How is the time-dependent change of magnetization described mathematically?

After this course you

1. Can describe the principle of MR detection and excitation
2. Can explain how MR excitation is frequency selective (resonance)
3. Understand the principle of relaxation to the equilibrium magnetization
4. Know what are the major relaxation times and how they phenomenologically affect magnetization in biological tissue, in particular that of water.
5. Can explain the elements of the Bloch equations and FID
6. Understand the MR contrast strongly depends on experimental parameters

What do we know about MR so far?

Need:
- Nucleus with non-zero spin
- Magnetic field $B_0$

Get:
- Nuclear (equilibrium) magnetization $M_0$
  (Magnitude dictated by Boltzmann distribution)
- $M_0$ increases with
  1. Number of spins in voxel
  2. Magnetic field $B_0$
  3. Gyromagnetic ratio $\gamma$

Imaging $^1H$ in $H_2O$ is most sensitive

Thermodynamic equilibrium magnetization $M_0$ is $\parallel B_0$

$$\frac{dM_0}{dt} = M_0 \times \gamma B_0 = 0$$
$M_0$ does not precess

All this does not generate a measurable signal
9-1. How is the MR signal detected?

**Faraday’s Law of Induction**

\[ \varepsilon = -\frac{d\Phi_s}{dt} \quad \text{Magnetic flux} \ \Phi_s \]

**Lenz’s Law**

Induced voltage \( \varepsilon \) \( \Rightarrow \) current \( \Rightarrow \) magnetic field opposes the change in the magnetic flux that produces the current (Completely analogous to power generation!)

**Biot-Savart Law**

Magnetic field falls off with \( r^{-2} \)

\[ \mathbf{dA} = \text{constant} \]

\[ \varepsilon = \mathbf{dB}/\mathbf{dt} \]

9-7

9-2. Rotating frame revisited

Equation of motion for \( \mathbf{M} \) (always valid in any reference frame) in presence of \( \mathbf{B}_0 \)

\[ \frac{d\mathbf{M}}{dt} = -\gamma \Delta \mathbf{B}^{\text{eff}} \times \mathbf{M} \]

**Rotating frame**: reference frame rotating about \( z \) at frequency \( \omega_{\text{RF}} \)

- **Case I**: non-rotating reference frame \( (\omega_{\text{RF}} = 0) \)
  - \( \Rightarrow \) magnetization *precesses* in \( xy \) plane with frequency \( \gamma B_0/2\pi \)

- **Case II**: rotating frame with \( \omega_{\text{RF}} = \omega_\perp \)
  - \( \Rightarrow \) magnetization is stationary (*precesses* in \( xy \) with zero frequency)
  - Equation of motion is still valid, i.e. precession frequency \( \gamma \Delta \mathbf{B}^{\text{eff}}/2\pi \)
  - \( \Rightarrow \Delta \mathbf{B}^{\text{eff}} = 0 \)

Larmor frequency \( \Omega \) in the rotating frame:

\[ \Omega = \gamma \Delta \mathbf{B}^{\text{eff}} \]

\[ \Delta \mathbf{B}^{\text{eff}} = \mathbf{B}_0 - \omega_{\text{RF}}/\gamma \]

Fund BioMag 2016
Supplement: Rotating frame

What are the quantum-mechanical equivalencies?

**Schrödinger representation:**

\[
\frac{ih}{d t} |\psi_s(t)\rangle = H_S |\psi_s(t)\rangle
\]

If \( H_S = \text{const} \) in \( t \):

\[
|\psi_s(t)\rangle = e^{-iH_st/h}
\]

**Interaction representation** (Higher order perturbation theory)

\[
|\psi_s(t)\rangle = e^{iH_{s,t}/h}|\psi_s(t)\rangle
\]

\[
\frac{d}{dt} |\psi_s(t)\rangle = V_s(t) |\psi_s(t)\rangle
\]

\[
V_s(t) = e^{iH_{s,t}/h} V_s(t) e^{-iH_{s,t}/h}
\]

**Quantum mechanical equivalencies:**

\[
M_z \propto \langle l_z \rangle, \quad M_x \propto \langle l_x \rangle, \quad M_y \propto \langle l_y \rangle
\]

For one spin-1/2 \((1H)\), i.e. two energy levels

\[
I_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad I_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}
\]

\[
\frac{d}{dt} |\psi_s(t)\rangle = [H_S^0 + V(t)] |\psi_s(t)\rangle
\]

For spin:

\[
H_S^0 = \hbar \gamma B_0 I_z
\]

\[
V(t) = \hbar \gamma I_x \left[ \cos(\omega_{RF} t) I_x + \sin(\omega_{RF} t) I_y \right]
\]

What is \( V(t) [\omega_{RF}=\gamma B_0] \)?

\[
V(t) = \hbar \gamma B_1 I_x
\]

**Quantum mechanical equivalencies:**

\[
B_0 \propto I_z, \quad B_{1xy} \propto I_{xy}
\]

9-3. What is the motion of magnetization when an RF field induces a flip angle?

**Laboratory frame** of reference

\[
\frac{d\vec{M}}{dt} = -\gamma (\vec{B}_1(t) + \vec{B}_0) \times \vec{M}
\]

**Rotating frame** of reference

\[
\frac{d\vec{M}}{dt} = -\gamma \vec{B}_1 \times \vec{M}
\]

\( B_1 \) radiofrequency field at Larmor frequency \( \omega_L \) applied in transverse \((xy)\) plane for duration \( \tau \)

\( \Rightarrow \) **nutation** (at \( \omega_L \)) of \( M \) as it tips away from the \( z \)-axis.

RF field rotates \( M \) towards \( xy \) plane

Amplitude \( B_1 \) determines how quickly the magnetization is rotated.

\[
\text{flip angle } \alpha = \gamma B_1 \tau \; \text{[rad]}
\]

\[
M_z = M_0 \cos \alpha
\]

\[
M_{xy} = M_0 \sin \alpha
\]

In MRI typically \( \gamma B_1 / 2\pi \sim 0.1-1 \text{kHz} \)

\( (\tau \sim 1 \text{ms}) \)
What is “resonance”?

What range of frequencies can be excited with a given RF pulse?

At $\Delta \omega = \omega_0 - \omega_{RF}$ (from $\omega_0$) magnetization experiences effective field strength $B_{eff}$

$$\gamma B_{eff} = \sqrt{\left(\gamma B_1\right)^2 + (\Delta \omega)^2}$$

Rotation axis : tilted by $\theta$.

“on resonance”:

$\gamma B_1 >> \Delta \omega$ $\rightarrow$ effective field $|| B_1$

$\Rightarrow$ short RF pulses ($t<1ms$)

RF field with amplitude $B_1$ can excite a range of frequencies on the order of $\pm \gamma B_1$

Quantum mechanical “resonance”

Transition probability highest : $h \nu = h\gamma B_0 / 2\pi$

9-4. How is the return to equilibrium $M_0$ governed?

Relaxation

Thermodynamic equilibrium

After excitation

Transverse magnetization:

(Along x and y-axis, on resonance)

$$\frac{dM_x(t)}{dt} = -\frac{M_x(t)}{T_1}$$
$$\frac{dM_y(t)}{dt} = -\frac{M_y(t)}{T_2}$$

Equations formally equivalent to linear attenuation coefficient (x-ray) (same solution)
What are the mechanisms of relaxation?

- **Tumbling of Molecule** (Brownian motion)
  - Creates local oscillating/fluctuating magnetic field

  Fluctuating magnetic field depends on orientation of the whole molecule & correlation time $\tau_c$ (=time for molecule to rotate 1 rad)

  Sources of fluctuating magnetic field:
  - Dipolar coupling between nuclei and solvent
  - Interaction between nuclear magnetic dipoles

Correlation function $G(t) \propto e^{-t/\tau_c}$

- Describes degree of correlation of motion $t$ sec apart

  Correlation time $\tau_c$:
  
  $$\tau_c = \frac{4\pi\eta r^3}{3kT}$$

  - $\eta$: viscosity
  - $k$: Boltzmann constant
  - $r$: size of molecule

What is the cause of loss of transverse Magnetization?

- **$T_2$: phenomenological time constant**
  - Range $10\mu$s (bone)… several s (water)
  - "transverse relaxation", "T2 relaxation"

- **Cause:**
  - Molecular dynamics and spin-spin interactions
  - Historically: "spin-spin" relaxation
  - Loss of signal in $xy$ plane
  - "Memory" relaxation

- **Rule of thumb for tissue water:**
  - The less "tissue" (bone, solutes, proteins, membranes) is in contact with bulk water, the longer bulk water $T_2$

After excitation:

$$M_{xy}(t) = M_{xy}(0)e^{-t/T_2}$$

Phase $\phi$ accrued over $\tau_c$:

$$\phi = \mathcal{B}_{\tau_c} \int_{voxel} \rho(\vec{r}) e^{i\mathcal{B}(\vec{r})\tau_c} dV \to 0$$

- $\tau_c$ large (immobile spins):
  - Large phase differences $\Rightarrow$ short $T_2$

  - Bone, membranes, proteins are MR-"invisible"
How does \( M_z \) return to equilibrium?

Longitudinal relaxation \( T_1 \)

After decay of \( M_{xy} \) by \( T_2 \): \( M_z \rightarrow M_0 \)

Longitudinal Relaxation (along z-axis)

\[
\frac{dM_z(t)}{dt} = \frac{M_z(t)-M_0}{T_1}
\]

\[
M_z(t) = M_0 \left(1 - e^{-t/T_1}\right) + M_z(0) e^{-t/T_1}
\]

Mechanisms: Incoherent molecular fluctuations on the order of the Larmor frequency \( \omega_0 \)
possibility of energy transfer \( \rightarrow \) matching frequency

Historically: spin-lattice relaxation
(heat lost to the surroundings)
\( T_1 \sim 0.5-5s \) (water)

Rule of thumb for water:
The less "tissue" is in contact with bulk water (bone, solutes, proteins, membranes), the longer bulk water \( T_1 \)

After \( T_2 \) relaxation:

\[
\frac{dM_z}{dt} = \gamma [M_y(t)B_x(t) - M_z(t)B_y(t)]
\]

\[
\frac{dM_x}{dt} = \gamma [M_z(t)B_y(t) - M_x(t)B_z(t)]
\]

\[
\frac{dM_y}{dt} = \gamma [M_z(t)B_x(t) - M_y(t)B_y(t)]
\]

- \( \gamma \vec{B} \times \vec{M} \)

Substituting \( \Omega = -\gamma B_0 + \omega_{RF} \) (\( B_0 = B_z \) is not time-dependent) yields:

Rotating reference frame

\[
\gamma [M_y(t)B_1^+(t) - M_x(t)B_y^+(t)]
\]

\[
-\Omega M_y(t) - \gamma M_z B_y^+(t)
\]

\[
\gamma M_z(t)B_1^+(t) + \Omega M_x
\]

Fund BioMag 2016

9-5. What equations describe the change in magnetization?

Bloch Equations

Add relaxation terms (\( T_1, T_2 \)) to the fundamental Eq of motion of magnetization:

Fund BioMag 2016
What characterizes the basic MR signal?

Free induction decay: Precession and relaxation (after RF pulse)

Transverse magnetization

\[ M_y(t) = M_{xy}(0)e^{-i\omega t}e^{-t/T_2} \]

Longitudinal magnetization

(after 90° RF excitation)

\[ M_z(t) = M_0(1 - e^{-t/T_1}) \]

How can T₁ changes be measured?

repetitive pulsing

Multipulse experiment with RF pulses applied every TR seconds

\[ M_z(t) = M_0(1 - e^{-t/T_1}) + M(0)e^{-t/T_1} \]

\[ M(0) = M_0 \cos \alpha \]

\[ \rightarrow M_z(t) = M_0(1 - e^{-t/T_1}) \]

The effect of T₁ (and T₂) on the signal depends on how it is measured

Optimal TR to detect changes in T₁?

Use noise error propagation calculation (Lesson 1)

\[ F = \max: \frac{\partial M_z(t)}{\partial T_1} = t \frac{M_0}{T_1^2}e^{-t/T_1} \]

\[ \frac{dF}{dT_1} = 0 = \frac{1}{T_1} e^{-t/T_1} - \frac{t}{T_1^2} e^{-t/T_1} \]

\[ t = \text{TR}_{\text{opt}} \cdot \frac{1}{T_1} \]

\[ \text{TR}_{\text{opt}} = \frac{T_1}{9-16} \]
## Summary

### Magnetic resonance so far

| Magnetic field $B_0$ | Equilibrium magnetization $M_0||z$ proportional to |
|----------------------|---------------------------------------------------|
|                      | 1. number of spins in voxel                       |
|                      | 2. Static magnetic field $B_0$                    |
|                      | 3. gyromagnetic ratio $\gamma$                   |
| RF field $B_1$       | tilts magnetization $M$ into transverse plane $xy$|
| (applied on-resonance i.e. $\omega_L$) | Precession of $M_{xy}$ is detected |
|                      | exponential decay of $M_{xy}$ exp. return of $M_z$ to $M_0$ |
|                      | reflect molecular environment                      |
|                      | source of contrast                                  |

1. Only mobile spins (e.g. water) are detected
2. $M_0$ reflects amount of nuclei and thus water content  
   [Water content varies 70-100ml/100g in body (poor contrast)]
3. Effect of $T_1$ and $T_2$ changes on image contrast depend strongly on experimental parameters (RF pulse timing and flip angle)